The Notion of Carried-Number, between the History of Calculating Instruments and Arithmetic

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This paper is based on a doctoral thesis about studying calculating instruments and deals with the very familiar primary school notion: the carried-number. We develop this notion in three ways: the history of calculating instruments and their mechanisation; a mathematical study of this notion within the place-value system; and an analysis of experimental data from an investigation of student and teacher understanding. For many students and teachers the notion of carried-number appears to be undeveloped mathematically.

This paper is part of a PhD research study about the teaching and learning of arithmetic in primary school, in particular year 6 students. An important aim was to bring together calculation and reasoning. The students studied different calculating instruments: the Chinese abacus, Napier's Bones and Genaille-Lucas' Rulers, and the slide rule. Through the study of those instruments, it was intended to develop understanding of the place-value system, the algorithms for calculating, and the notion of carried-number within the place-value system. In this paper we will focus on the notion of carried-number. Firstly, we show the link between the calculating instruments and the carried-number. Then, a mathematical analysis of the carried-number is given. Finally, there is an analysis of questionnaires given to students and to teachers about the notion of carried-number.

Calculating Instruments

Firstly, we will introduce some historical points with respect to calculating instruments. What calculating instruments were invented by humans before calculators? Marguin (1994) distinguishes between *calculating instruments* (for example, the abacus, Napier's Bones) and the *calculating machines*. Calculating machines differ from calculating instruments because they automate the carried-number. The mechanisation of the calculation is possible only thanks to the use of the place-value system. The first arithmetical machines that automated the carried-number, were made at the 17th century. The most well-known is the *Pascaline* or *Pascal's machine* built in 1642 in France by Blaise Pascal. However, a few years before, in 1623, Wilhelm Schickard had built the first machine in Germany.

To understand the history of calculating instruments and machines, we need to visualise in parallel the history of both mechanisation and numeration. The technological knowledge necessary for building those machines, which was like a mechanical clock, had been available in Europe since the end of the 13th century. Moreover, Indian numeration and calculation had been known in Europe since the 15th century (see Chabert, Barbin, & Guillemot, 1994; Guitel, 1975; Shärlig, 2001; Stévin, 1585). So, when the first machines were invented at the beginning of the 13th century, the knowledge necessary had been known for two centuries. For Marguin (1994, p. 42):

> Seuls des esprits hors du commun pouvaient […] transgresser le tabou pour s'aventurer dans la simulation mécanique d'un processus mental. Car il s'agissait bien de faire réaliser par une machine une opération de l'esprit, encore inaccessible au plus grand nombre.

Effectively, this represented a new way of thinking because machines could now replace humans for calculating. But we must not under-estimate the mathematical knowledge necessary to achieve the mechanisation of calculation. The use of a place-value system is essential, as well as the comprehension of calculating algorithms and in particular the notion of carried-number. At this time, Roman numbers and *tables à jetons* were commonly used. That means that a very high level of mathematics was essential to mechanise calculations. Of course, an environment with all technological skills available was also needed.

To understand what has been done in the 20th century to help humans to calculate, I refer to the French mathematician, Édouard Lucas (author of *Récréations mathématiques* in 1885). He wrote in 1891, in a book about theory of numbers, that multiplication and division of big numbers are "difficult problems" (p. 31). In one century there was a dazzling evolution, thanks to the generalisation of calculators and computers.

But where is the boundary between calculating instruments and calculating machines? The Chinese abacus was the first autonomous and portable calculating instrument. When taught to young children, they will later calculate quickly and surely. Their movements will become automated. For the most experienced users, calculating with the abacus can be faster than with a calculator. It is possible to name the couple {experienced user, abacus} as a machine. We can distinguish the *abacus-instrument* (with which the user can learn mathematics, in particular the place-value system and arithmetic algorithms) and the *abacus-machine* that is a machine that can calculate.

The Mathematical Study of the Carried-Number

The notion of carried-number was crucial to mechanise calculation. But, how can we define it mathematically?

The usefulness of a base 10 numeration system is to allow the representation of big numbers. This is difficult with the Roman numeration system, and also requires a large number of digits in binary. For example, let's write 1999 in different ways. In binary: $[1999]_{10} = [11\ 111\ 001\ 111]_2$ and with Roman numbers we have, depending on the period, MDCCCCLXXXXVIIII or MCMXCIX.

In a base system, a number's length is a function of its value. However, the most interest is the possibility of doing calculations quickly and easily.

As Lebesgue (1975, p. 9) reminds us, the invention of decimal numeration, maybe the most important of the history of sciences, is still not fully used at school:

> Notre enseignement n'utilise pas encore pleinement ce fait historique, le plus important peut-être de l'histoire des sciences : l'invention de la numération décimale.

In each rank of the decimal system (units, tens, hundreds, …) the digits from zero to nine can be written. A soon as ten is reached there is a transfer of numbers between ranks, that is to say 10 units = 1 ten, 10 tens = 1 hundred, 10 hundreds = 1 thousand, and so on. To do arithmetic operations we use this relation.

Definition: The carried-number enables us to manage the change of the place-value by making a transfer of numbers between ranks.

Let us look briefly at how carried-numbers are used to do addition, subtraction, and multiplication.

Addition

$$\begin{array}{rrrr} & {}^{1}5 & 3 & 8 \\ + & 1 & 9 & 1 \\ \hline & 7 & 2 & 9 \end{array}$$

At tens rank, we do the addition 9+3=12. And 12 tens = 120 that is two tens and one hundred. Thus, for the addition of two numbers, if the addition is at the units rank, the carried-number (if it exists) is one ten; at the tens rank the carried-number is one hundred; and so on. At rank i, the carried-number is one at rank (i+1). For the addition of three numbers, the maximum carried-number is two, because for the maximum 9×3=27, and if there is a carried-number at lower rank, the maximum is 27+2=29. For the addition of four numbers, the maximum carried-number is three, as 9×4=36 plus the maximum of three carried from the lower rank is 39.

Generalisation: To add m numbers at rank i, the carried-number is maximum (m-1) at rank (i+1), even if there is already a carried number at the lower rank.

To build a machine like the Pascaline, we need to know the possible carried-numbers. To take a carried-number into account, one more notch of the gears must be activated at the upper rank. The mechanisation is possible because it can be a maximum of one. The Pascaline can work in base ten and also twelve, and was still used in the 17^{th} century in France. We showed (Poisard, 2005a) that for any base, the maximum carried-number for the addition of two numbers is one. This machine makes multiplications by repeated additions. In fact, Napier's Bones, invented by John Napier in 1617 in Scotland, are more efficient for multiplication. Napier's Bones were used until the second part of the 19^{th} century in Europe.

Subtraction

$$\begin{array}{rr} 8 & {}_{1}1 \\ 1_{1} & 7 \\ \hline 6 & 4 \end{array}$$

We want to subtract seven units, but we only have one. For the subtraction we use two carried-numbers. At the units rank, ten units are added and at tens rank we take off one ten, that is to say 81_17=81+10_(17+10). In the units, we therefore do 11_7=4 and in the tens 8_(1+1)=6.

Generalisation: To subtract two numbers x and y (non null), the property used to do the calculation is: $x–y= x+10^{i+1} – (y+10^{i+1})$. That means 10^{i+1} is written as ten at rank i and one at rank (i+1).

Multiplication

$$\begin{array}{rrrrr} & 2 & 3 & 5 & \\ \times & & & 7 & \cancel{3} \\ \hline & 16 & 4 & 5 & \cancel{2} \end{array}$$

At the units rank, we obtain 5×7=35 therefore five units and three tens, the carried-number is three in the tens. After, 7×3=21, with the carried-number of three, 24 in the tens therefore four tens and two hundreds, that is a carried-number of two in the hundreds. Finally, 7×2=14 and the carried number of two, 16 in the hundreds.

Generalisation: To multiply two numbers at rank i, the carried-number is maximum eight at rank (i+1). Indeed, 9×9=81 and if there is a carried-number (maximum 8) at a lower rank 81+8=89.

This question is central to the study of Genaille-Lucas' Rulers invented in 1885 by Henri Genaille and Édouard Lucas in France. The rulers manage the carried-number for a multiplication by a number of one digit, for example 567 times 9. To build the rulers we need to know the carried-number possible for a multiplication. It depends on the multiplier, for example for the multiplier two the maximum carried-number is one (2×9=18), for the multiplier three it is two (3×9=27), …, for the multiplier six it is five (6×9=54),…, and so on. The maximum carried-number does not change if there is already one at a lower rank. On the Rulers, a system of triangles allows us to change a digit in an upper rank if necessary. Thus for 78×7=56+490=546, the carried-number is one hundred that changes 400 to 500. The Rulers manage this transfer.

Understanding of the Carried-Number

Students

The study took place at a centre in France that receives primary school students with their teacher for short periods. The aim of the centre is to make and study scientific objects. The workshop that is part of this study is about making and studying calculating instruments: the Chinese abacus, Napier's Bones and Genaille-Lucas' Rulers, and the slide rule.

We have two main research questions. First, what do students think a carried-number is? Second, will a workshop about making and studying calculating instruments be a good way to study this notion? Five classes at year 6 from three different schools in Marseille were asked to answer questionnaires.

- Class 1 from school A that came to the centre in October 2003.
- Class 2 from school B that came to the centre in December 2003.
- Class 3 from school A that came to the centre in January 2004.
- Class 4 from school C that came to the centre in April 2004.
- Class 5 from school C that came to the centre in April 2005, this was the control class and participated in a workshop on electronics, not calculating instruments. It had the same teacher as the class 4.

These questionnaires were completed at school, a few days before the beginning of the workshop and few days after the end. We ensured that students understood the questionnaires were anonymous. We collected 129 questionnaires for the pre-test (60 girls and 69 boys) and 127 for the post-test (62 girls and 65 boys).

The questionnaires consisted of five and seven questions (pre-test and post-test). Here, we will focus on only one question: "What is a carried-number?" This question follows one about a calculation, the multiplication 632×73 which was asked to be done in columns.

In French, the carried-number is *la retenue*, that is, something kept or remembered, we could translate it as *retainer*. Another meaning of *retenue* is a detention when you are kept in at school. Few students gave the second meaning as an answer.

The first thing about the results as a whole is that the answers very vague and without mathematical meaning at the pre-test, but became more specific at the post-test (figure 1). Answers of the type "something kept or remembered" (20.2 % then 14.2 %) or "something too many" (10.9 % then 6.3 %) decreased in favour of "a ten" (17.8 % then 30.7 %) and "something in the next column" (6.2 % then 15.7 %). Thus, for the post-test the answer is often limited to a specific case that corresponds to the question: "For the addition of two numbers, what is the carried-number got by the addition in the units?" And the answer is: "a ten" and for some "a ten in the next column" (3.9 % then 6.3 %).

In the students' data, we classified as arithmetical answers the following five items: "a ten"; "a number of two digits"; "something that overtakes 9"; "the notion of passage between the units and the tens"; or "the passage to the next column". This classification represents 27.9 % of the students in the pre-test and 47.2 % in the post-test (table 1).

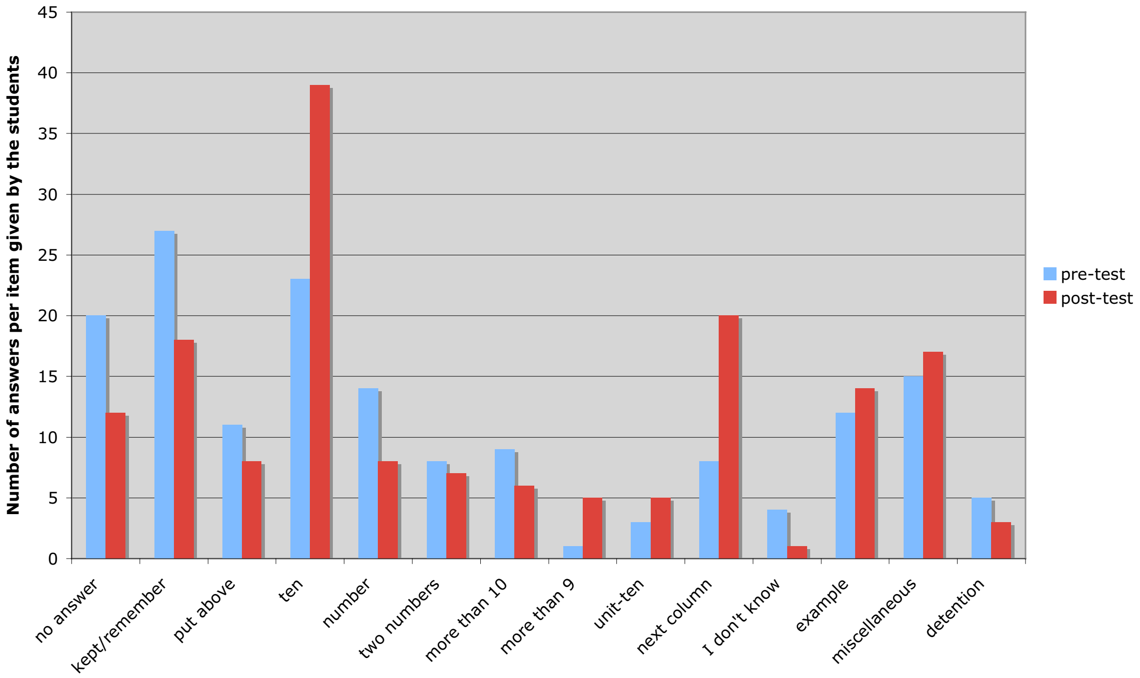

Figure 1. Number of answers per item for the total.

How can we explain these results? We analyse three possibilities. Is this change due to the workshop? Is it due to a change in the didactical contract? Could it be due to the personal development of mathematical understanding of the students?

Table 1
Evolution of Arithmetical Answers for the Five Classes

Arithmetical answers	Class 1	Class 2	Class 3	Class 4	Class 5	Total
Pre-test	28.0 % (7/25)	24.1 % (7/29)	22.7 % (5/22)	32.1 % (9/28)	32.0 % (8/25)	27.9 % (36/129)
Post-test	66.7 % (16/24)	37.5 % (9/24)	41.7 % (10/24)	39.3 % (11/28)	51.9 % (14/27)	47.2 % (60/127)

Firstly, it could be argued that the calculating instrument workshop did not contribute to the evolution of understanding about carried-number since the control group is not distinguished from the others. But why did all the classes increase over the period of the workshop, whatever time of the year that took place? There is no difference between the pre-test taken in September 2003 and April 2004. That means something happened at the time of the workshop.

The two classes with the best results in pre-test had the same teacher (classes 4 and 5). Note that class 5 is the control group. Perhaps the teaching of this teacher influences the good results, even of the control class. It is possible, in addition, that this teacher changed his methods after the class 4 workshop, that could have contributed to the very good results of the control class, that took place a year later.

Secondly, another possible explanation for the change might be a change in didactical contract, and not engagement with mathematical knowledge. The concept of didactical contract (Brousseau, 1998; Douady & Mercier, 1992) sheds light on the relationship between the teacher and the students, that is to say on all the explicit and implicit rules between them. All the questionnaires were administered at school with both teacher and researcher present. Do students learn to write answers that are closer to some mathematical preoccupation? Thus participation in the study, not the workshop itself, could be responsible for the change. However, if this was so, we should also see a difference between the classes in September who were just beginning to know their new teacher, and those in April. No such difference exists, so there is no evidence that it is a question of didactical contract.

Finally, it is possible that the evolution of their thinking about carried-number comes from the personal reflection of the students during the month between the pre- and post-test, either alone or in class. That is, it is not the directly the participation in the workshop that creates change, it is the ability of students themselves to think about the questions given in the pre-test. We have no evidence either way for this suggestion.

It is likely, therefore, that there is some combination of effect from the workshop itself, teacher influence, and personal student development. So, what is the overall influence of the workshop? The answer is still difficult to give. A possible conclusion is that the simple fact of asking the question: “What is a carried-number?” gives students something to think about. This question is rarely so directly raised, and its effect deserves more attention. The evidence shows that the workshop is not an obstacle to the study of carried-number. We continue to think that it is a good way to study this notion, but this is still unproven.

Teachers

After the analysis of the students’ questionnaires, we needed to know more about the possible answers that a primary teacher might give to the same questions. In particular, we wanted to know what might be their answer to the question: “What is a carried-number?” Thus, we questioned teachers in in-service education during the spring of 2005 in Marseille.

We collected 49 questionnaires, from 41 women and seven men. Firstly, three teachers could not give any answer to the question. Only one teacher envisaged the case of subtraction, explaining that it is a “contrivance”. The carried-number is therefore only envisaged for the addition, even though the teachers questioned teach subtraction and multiplication. Some interesting notions appeared, like “a new group” (15/46 teachers) or “the upper order” (10/46) (see figure 2 for full results). For 19 teachers, the carried-number is “a ten, or a hundred or etc.”, but ten only mention “the ten” and/or “the ten and the unit”, not indicating awareness of all the cases. It was surprising that three teachers responded “something kept or remembered”, eleven teachers “the left column” and one teacher answered “>10” which is mathematically incorrect. The notion of “place-value system”, closely linked to the definition of the carried-number, is cited by five teachers (all of whom were teachers at senior primary level).

In the teachers’ data, we classified as arithmetical answers the following six items: “a ten, or a hundred or etc.”; “something that overtakes 9”; “notion of passage”; “new group, new unit or ten”; “notion of decimal system or place-value system”; or “notion of upper order”. The notions of “left column” or “exchange or transfer” were deemed to be not satisfactory because they are not relevant as answers for students. Thus, more than two thirds of the teachers who gave an answer (31/46) had relevant answers.

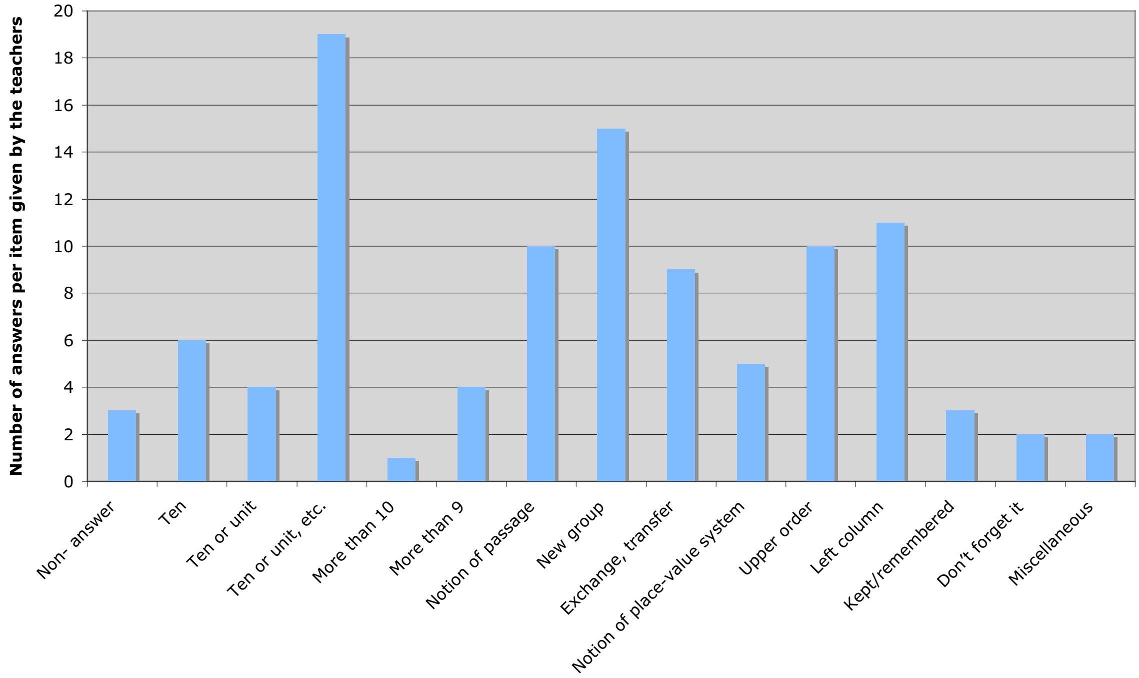

Figure 2. Definition of carried-number given by the teachers.

There appears not to be universal understanding of the notion amongst practising teachers. Thus, the question of explaining carried-number deserves to be asked of teachers. The answer is not immediate and requires reflection. Such an explanation does not refer to new knowledge but mobilises and reorganises knowledge relative to place-value system. This question is relevant for teacher education.

Conclusion

We showed that the notion of carried-number is not easily defined by students, nor by teachers. Here is a quotation from Clara, a student in year 6:

> A carried-number is a digit that we add to a digit when in the result we find a number instead of a digit, we put the ten to the after digit and we find the result.

This answer could make us smile. But, effectively, the problem lies in the fact that, when we do an arithmetical operation, we sometimes obtain a number with more than one digit and this cannot be placed in one rank. Furthermore the value of these digits depends on the rank.

This question necessitates a real reflection in mathematics classes. Its study requires in depth comprehension of the place-value system. As for primary school, as for teacher education: teachers also need to confront this problem in their preparation.

To work on this difficult notion in class, we propose studying calculating instruments (see Hébert, 2004; Bartolini Bussi, 2000; Uttal, Scudder, & Deloache, 1997; Bosch & Chevallard, 1999; Poisard, 2005a). In particular, the Chinese abacus allows us to write up to fifteen in each place-value, and then we can make changes between place-value *with the* *hand*. I have shown that the study, from primary school to the university, of the Chinese abacus (Poisard, 2005b) and asking students how it works is a relevant way to investigate the place-value system, calculation algorithms, and the notion of carried-number.

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